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A BPM SOLUTION FOR ELLIPTIC PLATES SUBJECTED TO ECCENTRIC LOADS†

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Abstract—A higher-order boundary perturbation method (BPM), formulated to treat a class of problems defined in an elliptic domain, is developed to obtain the Green's function due to an eccentric source. The method, based on a dual perturbation, leads to expansion solutions expressed in terms of ellipticity and eccentricity perturbation parameters. General explicit expressions for equivalent boundary conditions on the perturbed boundary are first derived to treat the class of problems for which the associated boundary conditions are of the Dirichlet or Neumann type. The BPM is applied to investigate a clamped elliptic plate subject to eccentric loads. Estimates of the accuracy of the method are given. The BPM is seen to yield reasonably accurate solutions for moderately elliptic domains and moderate ellipticities.

1. INTRODUCTION

In this paper, an analytic solution for the bending of a plate of moderate ellipticity, clamped along the boundary and subjected to a lateral force applied eccentrically with respect to the centre is considered and obtained.

While plates having relatively simple geometric shapes (rectangular, circular, etc.) have been analysed and solutions obtained for a variety of loading cases and support conditions, for shapes other than these, one usually turns to approximate or numerical methods. Among the latter, finite element methods have been extensively used and often yield important and useful results. However, finite element techniques do not have the capability of establishing analytic expressions which express a trend in the behaviour. On the other hand, the method developed here leads to expressions which yield the response for bodies of varying geometries and therefore represents a significant advantage over numerical methods which require a complete recalculation for each specific geometry considered.

It appears that the only cases of bending of elliptic plates which have been treated analytically are those of uniformly loaded plates: an exact solution for the clamped plate, obtained by Bryan, is given by Love[1]. The corresponding solution for a simply supported plate was given by Galerkin[2]. Other loading cases apparently have not been treated since the solutions usually require resorting to elliptic coordinates (with the ensuing complexity in the higher-order boundary conditions) or the use of Mathieu functions the numerical evaluation of which presents considerable difficulties. Indeed, from a perusal of the literature, it appears that even the symmetric case of a centrally loaded elliptic plate has not been investigated.

The more complex problem considered here, an eccentrically loaded elliptic plate, is not amenable to a direct and tractable analytic solution for all ellipticities and eccentricities. However, for small to moderate ellipticities and eccentricities, the problem may be treated by a boundary perturbation method (BPM).

Recently, Parnes and Beltzer[3] developed a BPM for the separate treatment of systems existing in an elliptic domain as well as for systems defined in circular domains and subject to eccentric forces. While the development of Ref. [3] does not treat the combination of ellipticity and eccentricity, it does provide a basis and preliminary expressions for the investigation of the present problem using a higher-order BPM scheme. Some aspects of the BPM have been studied by Parnes[4] where it was shown that a higher-order BPM leads

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to substantially accurate results for moderate ellipticities or eccentricities and yields an upper bound for the stiffness of elastic systems.

Following the methodology of Ref. [3], the general expressions for a second-order scheme which permit a resolution of the problem at hand is developed first in Section 2. The treatment of this section is quite general and is not confined to plates: general two-dimensional problems are considered for a Green's function in an elliptic domain, for which the associated boundary conditions are of the Dirichlet or Neumann type. Appropriate transformations and expressions are derived by means of a dual perturbation, leading to the required equivalent boundary conditions on the perturbed curves.

In Section 3, the plate problem is treated as a particular case of the general development. It can be noted that once the basic expressions of the previous section have been established, the solution, while requiring considerable algebraic manipulations, leads to simple expressions dependent on the non-dimensional ellipticity and eccentricity perturbation parameters.

In Section 4, numerical results are presented for the plate displacements along the principal axes. The effects of ellipticity and eccentricity are analysed. The results obtained by the BPM are compared with a finite element solution in order to provide an indication of the accuracy.

2. GENERAL BPM EXPRESSIONS FOR ECCENTRIC SOURCES IN AN ELLIPTIC DOMAIN

An elliptic domain is considered, bounded by a curve C_ε with semi-major and semi-minor axes a and b, respectively, and lying in the x–y plane with center $\bar{0}$. The ellipticity is defined by

$$\varepsilon = a/b - 1. \tag{1}$$

The governing differential equation for the Green's function due to a load (or source) of strength P acting at a point 0, having eccentricity l with respect to $\bar{0}$, is

$$L[f(r,\theta)] = P\delta(r) \tag{2}$$

where (r,θ) is the polar coordinate sysem with origin at 0 (Fig. 1), L the linear differential operator, and δ the Dirac-delta function.

The appropriate boundary conditions considered here are prescribed as

$$f|_{C_\varepsilon} = f_0 \quad \text{and/or} \quad \left.\frac{\mathrm{d}f}{\mathrm{d}n_\varepsilon}\right|_{C_\varepsilon} = f_n \tag{3a, b}$$

where f_0 and f_n are known quantities and n_ε is normal to C_ε at all points.

Clearly the problem, as posed by eqns (2) and (3) is not amenable to a direct analytic solution. If it is assumed however, that $f(r,\theta)$ is analytic throughout the x–y plane, then solutions may be found by considering the curve C_ε to be a perturbation of a circular curve

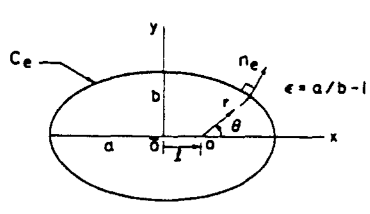


Fig. 1. Geometry of the problem.

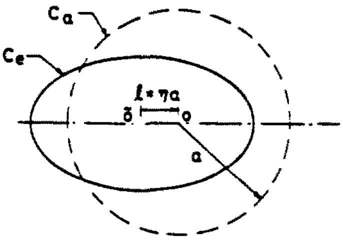


Fig. 2.

C_a, centred about point 0 (Fig. 2), on which appropriate equivalent boundary conditions are prescribed, i.e. one resorts to a BPM. The perturbation relations between the two curves depend on the ellipticity ε defined by eqn (1) and on the eccentricity of 0 with respect to $\bar{0}$, namely

$$\eta = \frac{l}{a}. \tag{4}$$

Following generally developed methods of perturbation theory, $f(r,\theta)$ is expanded in a power series in the perturbation parameters ε, η

$$f(r,\theta) = \sum_{i=0}^{N} \sum_{j=0}^{N} \eta^i \varepsilon^j f^{(i,j)}(r,\theta). \tag{5}$$

Substituting in eqn (2), using the linearity property of the differential operator L, and noting that the relation obtained must be satisfied for all small $\varepsilon > 0$, $\eta > 0$, yields

$$L[f^{(0,0)}] = P\delta(r) \tag{6a}$$

and

$$L[f^{(i,j)}] = 0\,; \quad i,j = 0,1,2,\ldots,N; \quad i+j > 0. \tag{6b}$$

It is noted that eqns (6) represent a sequence of equations which must be satisfied within the domain bounded by C_a and $f^{(i,j)}$ must satisfy appropriate boundary conditions on C_a. Attention is now turned to these boundary conditions.

First a polar coordinate system $(\bar{r},\bar{\theta})$ is defined with origin at $\bar{0}$, the centre of the ellipse. A generic point P_e on C_e is defined as having coordinates $(r_e,\bar{\theta})$ where $r_e(\bar{\theta})$ is the variable radial distance from $\bar{0}$ (Fig. 3). Curve C_e may then be considered to be a perturbation of the circumscribing circle C_0 of radius a, with points P_e on C_e being perturbed points of P_0 on C_0. Symbolically, the perturbation relation $C_0 \rightarrow C_e$ may be written as

$$a \rightarrow r_e = r_e(a,\bar{\theta},\varepsilon). \tag{7}$$

The relation between the two circles of identical radius a, C_0 and C_a, is now considered

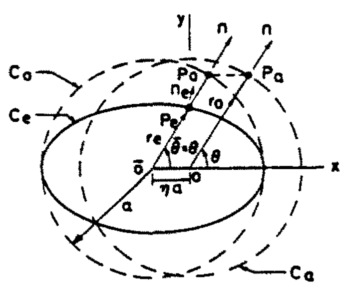


Fig. 3. Dually perturbed curves.

where the eccentricity parameter η is not large. Curve C_0 may then, in turn, be considered as a perturbation of curve C_a with points P_0 on C_0 being perturbed points of P_a on C_a.

Symbolically, the perturbed relation $C_a \to C_0$ may be written as

$$a \to r_0 = r_0(a, \theta, \eta) \tag{8}$$

where r_0 represents the variable radial distance from 0 to curve C_0. It is noted, from Fig. 3, that P_0 and P_a have the same coordinates, namely $\bar{r} = r = a$ and $\bar{\theta} = \theta$ in the respective systems. Thus, by means of the double mapping

$$P_a(r = a, \theta) \to P_0(\bar{r} = a, \bar{\theta} = \theta) \to P_e(\bar{r} = r_e, \bar{\theta} = \theta).$$

Letting n represent the normal to curves C_0 and C_a, it can be noted too that this normal is in the same direction for both curves. The transformation of a given function f, $f(P_0) \to f(P_a)$ as well as that for $f_{,n}(P_0)$ is now sought. In what follows, expansions will be performed to order $N = 2$. Assuming that f is analytic, and observing that P_0 and P_a possess the same y-coordinate, a Taylor series expansion yields

$$f|_{P_0} = f|_{P_a} - \left.\frac{\partial f}{\partial x}\right|_{P_a}(\eta a) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{P_a}(\eta a)^2. \tag{9}$$

Now

$$\left.\frac{\partial f}{\partial x}\right|_{P_a} = \left.\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\right|_{P_a} + \left.\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right|_{P_a}. \tag{10}$$

Furthermore, since

$$\frac{\partial r}{\partial x} = \cos\theta \tag{11a}$$

$$\frac{\partial r}{\partial y} = \sin\theta \tag{11b}$$

and noting that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{11c}$$

one obtains

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \tag{12}$$

Substituting in eqn (10)

$$\frac{\partial f}{\partial x} = \cos\theta\, f_{,r} - \frac{\sin\theta}{r} f_{,\theta} \tag{13a†}$$

and hence, on the circle C_a

† Here, and in all subsequent expressions, derivatives with respect to a variable are denoted by a subscript preceded by a comma, e.g. $f_{,r} \equiv \partial f/\partial r$, etc.

$$\left.\frac{\partial f}{\partial x}\right|_{P_a} = \left.\left(\cos\theta\, f_{,r} - \frac{\sin\theta}{a} f_{,\theta}\right)\right|_{P_a}. \tag{13b}$$

Similarly

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{P_a} = \left[\cos^2\theta\, f_{,rr} - \frac{\sin 2\theta}{a} f_{,\theta r} + \frac{\sin^2\theta}{a} f_{,r} + \frac{\sin^2\theta}{a^2} f_{,\theta\theta} + \frac{\sin 2\theta}{a^2} f_{,\theta}\right]_{P_a}. \tag{14}$$

Equation (9) can then be written as

$$f|_{C_0} = f|_{C_a} - a\eta D_x[f]|_{C_a} + \frac{a^2\eta^2}{2} D_{xx}[f]|_{C_a} \tag{15}$$

where D_x and D_{xx} are considered as differential operators:

$$D_x = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{16a}$$

$$D_{xx} = \cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{a}\frac{\partial}{\partial r} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2} + \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial\theta} \tag{16b}$$

which transform the function $f|_{C_a}$ to $f|_{C_0}$.

Substituting eqn (5) in the right-hand side of eqn (15) one obtains

$$f|_{C_0} = F_0|_{C_a} - aF_1|_{C_a} + \frac{a^2}{2}F_2|_{C_a} \tag{17}$$

where

$$F_0|_{C_a} = f^{(0,0)}|_{C_a} \tag{18a}$$

$$F_1|_{C_a} = \sum_{i=0}^{1}\sum_{j=0}^{2}\eta^{i+1}\varepsilon^j D_x[f^{(i,j)}(a,\theta)]_{C_a} \tag{18b}$$

$$F_2|_{C_a} = \sum_{j=0}^{2}\eta^2 D_{xx}[f^{(0,j)}(a,\theta)]_{C_a}. \tag{18c}$$

Expanding and collecting like powers in η and ε

$$f|_{C_0} = G^{(0)} + \varepsilon G^{(1)} + \varepsilon^2 G^{(2)}|_{C_a} \tag{19}$$

where

$$G^{(j)}(\eta) = g^{(0,j)} + g^{(1,j)}\eta + g^{(2,j)}\eta^2 \tag{20}$$

with

$$\left.\begin{aligned} g^{(0,j)} &= f^{(0,j)} && (21a)\\ g^{(1,j)} &= f^{(1,j)} - aD_x^{(0,j)} && (21b)\\ g^{(2,j)} &= f^{(2,j)} - aD_x^{(1,j)} + \frac{a^2}{2}D_{xx}^{(0,j)}. && (21c)\end{aligned}\right\}\ (j = 0, 1, 2)$$

In the above, the operators $D^{(i,j)}$ indicate that the operation is to be performed on the function $f^{(i,j)}$, i.e. $D_x^{(i,j)} \equiv D_x[f^{(i,j)}]$, etc. It is to be recalled that the expressions on the right-hand side of eqns (19)–(21) are evaluated on C_a, i.e. at $r = a, \theta$.

For the case of the normal derivative f_n at P_0, it is clear that eqns (19)–(21) remain the same with $f^{(i,j)}$ replaced by $f_n^{(i,j)}$ everywhere, since, as has been noted, the normal direction on C_0 is in the same direction as on C_a.

Having derived the transformation $f|_{C_a} \to f|_{C_0}$, a transformation is required from C_0 to C_ε both for f and f_n. This transformation, dependent on the parameter ε, and given for a function of the form

$$\tilde{F}|_{C_0} = \sum_{j=0}^{2} \varepsilon^j \tilde{G}^{(j)}(r,\theta)|_{C_0} \tag{22}$$

was derived in Ref. [3]

$$\tilde{F}|_{C_\varepsilon} = \tilde{G}^{(0)} + \{\tilde{G}^{(1)} + {}_0\Psi_1[\tilde{G}^{(0)}]\}\varepsilon + \{\tilde{G}^{(2)} + {}_0\Psi_1[\tilde{G}^{(1)}] + {}_0\Psi_2[\tilde{G}^{(0)}]\}\varepsilon^2|_{C_0}. \tag{23}$$

In the above ${}_0\Psi_i$ are differential operators acting on functions $\tilde{G}^{(j)}$, given by

$${}_0\Psi_1 = -a\sin^2\theta \frac{\partial}{\partial r} \tag{24a}$$

$${}_0\Psi_2 = \frac{a}{2}\sin^2\theta\left[(2\sin^2\theta - \cos^2\theta)\frac{\partial}{\partial r} + a\sin^2\theta\frac{\partial^2}{\partial r^2}\right]. \tag{24b}$$

It can be noted that for the problem at hand, upon letting $\tilde{G}^{(j)}(r,\theta)|_{C_0} \equiv G^{(j)}(r,\theta,\eta)|_{C_0}$, it follows that $\tilde{F}|_{C_\varepsilon} \to f|_{C_\varepsilon}$. Substituting eqns (20) and (21) in eqn (23) and collecting again like powers in η and ε, one obtains

$$f|_{C_\varepsilon} = \sum_{i=0}^{2} [g^{(i,0)} + \{g^{(i,1)} + {}_0\Psi_1[g^{(i,0)}]\}\varepsilon + \{g^{(i,2)} + {}_0\Psi_1[g^{(i,1)}] + {}_0\Psi_2[g^{(i,0)}]\}\varepsilon^2]\eta^i. \tag{25}$$

The transformation of f_n on C_0 to the normal on C_ε proceeds in a similar manner. For a function of the form given by eqn (22), the transformation, derived in Ref. [3] was established:

$$\left.\frac{\partial\tilde{F}}{\partial n_\varepsilon}\right|_{C_\varepsilon} = \tilde{G}_{,r}^{(0)} + \{\tilde{G}_{,r}^{(1)} + {}_n\Psi_1[\tilde{G}^{(0)}]\}\varepsilon + \{\tilde{G}_{,r}^{(2)} + {}_n\Psi_1[\tilde{G}^{(1)}] + {}_n\Psi_2[\tilde{G}^{(0)}]\}\varepsilon^2 \tag{26}$$

where the differential operators ${}_n\Psi_i$ are

$${}_n\Psi_1 = -a\sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin 2\theta}{a}\frac{\partial}{\partial\theta} \tag{27a}$$

$${}_n\Psi_2 = \frac{a^2}{2}\sin^4\theta\frac{\partial^3}{\partial r^3} + \frac{a}{2}(2\sin^2\theta - \cos^2\theta)\sin^2\theta\frac{\partial^2}{\partial r^2}$$
$$-\tfrac{1}{2}(\sin 2\theta)^2\frac{\partial}{\partial r} - \sin^2\theta\sin 2\theta\frac{\partial^2}{\partial r\,\partial\theta} + \frac{\sin 4\theta}{4a}\frac{\partial}{\partial\theta}. \tag{27b}$$

Proceeding as in the case of $f|_{C_\varepsilon}$ one obtains

$$\left.\frac{\partial f}{\partial n_\varepsilon}\right|_{C_\varepsilon} = \sum_{i=0}^{2} [g_{,r}^{(i,0)} + \{g_{,r}^{(i,1)} + {}_n\Psi_1[g^{(i,0)}]\}\varepsilon + \{g_{,r}^{(i,2)} + {}_n\Psi_1[g^{(i,1)}] + {}_n\Psi_2[g^{(i,0)}]\}\varepsilon^2]\eta^i. \tag{28}$$

Noting from eqn (21a) that $g^{(0,0)} = f^{(0,0)}$, $g_{,r}^{(0,0)} = f_{,r}^{(0,0)}$, and setting

$$f_0 = f^{(0,0)}|_{C_a} \tag{29a}$$

$$f_n = f^{(0,0)}_{,r}|_{C_a} \tag{29b}$$

eqn (25) is satisfied for all η and ε provided

$$\left.\begin{aligned} &g^{(i,0)} = 0 && i > 0 \\ &\left.\begin{aligned} &g^{(i,1)} + {}_0\Psi_1[g^{(i,0)}] = 0 \\ &g^{(i,2)} + {}_0\Psi_1[g^{(i,1)}] + {}_0\Psi_2[g^{(i,0)}] = 0. \end{aligned}\right\} && i \geqslant 0 \end{aligned}\right\} \tag{30}$$

Similarly, eqn (28) is satisfied provided

$$\left.\begin{aligned} &g^{(i,0)}_{,r} = 0 && i > 0 \\ &\left.\begin{aligned} &g^{(i,1)}_{,r} + {}_n\Psi_1[g^{(i,0)}] = 0 \\ &g^{(i,2)} + {}_n\Psi_1[g^{(i,1)}] + {}_n\Psi_2[g^{(i,0)}] = 0. \end{aligned}\right\} && i \geqslant 0 \end{aligned}\right\} \tag{31}$$

Equations (30) and (31) thus represent the required boundary conditions on C_a. Substituting the definitions of eqns (21) and the operators given by eqns (16) and eqns (24) and (27), leads to the following explicit conditions on $f^{(i,j)}|_{C_a}$ and $f^{(i,j)}_{,r}|_{C_a}$, $(i+j \geqslant 1;\ i \leqslant 1,\ j \leqslant 1)$:

$$f^{(0,1)} = -{}_0\Psi_1[f^{(0,0)}] = a \sin^2\theta\, f^{(0,0)}_{,r} \tag{32a}$$

$$f^{(1,0)} = aD^{(0,0)}_x = a\left[\cos\theta\, f^{(0,0)}_{,r} - \frac{\sin\theta}{a} f^{(0,0)}_{,\theta}\right] \tag{32b}$$

$$\begin{aligned} f^{(1,1)} &= aD^{(0,1)}_x + {}_0\Psi_1[aD^{(0,0)}_x - f^{(1,0)}] \\ &= a\left(\cos\theta\, f^{(0,1)}_{,r} - \frac{\sin\theta}{a} f^{(0,1)}_{,\theta}\right) \\ &\quad - a\sin^2\theta\left[a\cos\theta\, f^{(0,0)}_{,rr} - \sin\theta\left(f^{(0,0)}_{,\theta r} - \frac{1}{a} f^{(1,0)}_{,\theta}\right) - f^{(1,0)}_{,r}\right] \end{aligned} \tag{32c}$$

$$f^{(0,1)}_{,r} = -{}_n\Psi_1[f^{(0,0)}] = a\sin^2\theta\, f^{(0,0)}_{,rr} - \frac{\sin 2\theta}{a} f^{(0,0)}_{,\theta} \tag{33a}$$

$$f^{(1,0)}_{,r} = aD^{(0,0)}_{x,r} = a\left[\cos\theta\, f^{(0,0)}_{,rr} - \frac{\sin\theta}{a}\left(f^{(0,0)}_{,r\theta} - \frac{1}{a} f^{(0,0)}_{,\theta}\right)\right] \tag{33b}$$

$$\begin{aligned} f^{(1,1)}_{,r} &= aD^{(0,1)}_{x,r} - {}_n\Psi_1[f^{(1,0)} - aD^{(0,0)}_x] \\ &= a\cos\theta\, f^{(0,1)}_{,rr} - \sin\theta\left(f^{(0,1)}_{,\theta r} - \frac{1}{a} f^{(0,1)}_{,\theta}\right) \\ &\quad + a\sin^2\theta\left[f^{(1,0)}_{,rr} - a\cos\theta\, f^{(0,0)}_{,rrr} + \sin\theta\left(f^{(0,0)}_{,rr\theta} - \frac{2}{a} f^{(0,0)}_{,\theta r} + \frac{2}{a^2} f^{(0,0)}_{,\theta}\right)\right] \\ &\quad + \sin 2\theta\left[-\frac{f^{(1,0)}_{,\theta}}{a} + \cos\theta\, f^{(0,0)}_{,r\theta} - \frac{\sin\theta}{a} f^{(0,0)}_{,\theta\theta} - \sin\theta\, f^{(0,0)}_{,r} - \frac{\cos\theta}{a} f^{(0,0)}_{,\theta}\right]. \end{aligned} \tag{33c}$$

Explicit higher-order terms obtained from eqns (30) and (31) for $1 < i+j \leq 4$ ($i \leq 2$, $j \leq 2$) are given in Ref. [7].

The formulation of the boundary perturbation method, as derived above, is thus complete. To summarize, the solution to the boundary value problem

$$L[f^{(0,0)}] = P\delta(r) \tag{34a}$$

subject to the boundary conditions on C_a

$$f^{(0,0)}|_{C_a} = f|_{C_e} \quad \text{and/or} \quad f^{(0,0)}_{,r}|_{C_a} = \left.\frac{\mathrm{d}f}{\mathrm{d}n_e}\right|_{C_e} \tag{34b}$$

is first obtained. It is observed that this is often the known solution to an axi-symmetric problem in a circular domain. The method proceeds by solving sequentially a set of homogeneous equations

$$L[f^{(i,j)}] = 0, \quad i+j > 0 \tag{35}$$

subject to the appropriate boundary condition on C_a given by eqns (32) and (33) and the higher-order expressions presented in Ref. [7]. It is noted that since these individual problems are solved sequentially, the given boundary conditions on C_a for any $f^{(i,j)}$ are always defined, being dependent on $f^{(k,l)}$, $k \leq i$, $l \leq j$, with $k+l < i+j$.

3. GENERAL BPM SOLUTION FOR CLAMPED ELLIPTIC PLATE SUBJECTED TO ECCENTRIC LOADING

3.1. *Formulation and the $i = j = 0$ case*

Bending of an elastic elliptic plate of thickness h with semi-major and semi-minor radii a and b, respectively, and clamped along the boundary C_e is considered. The material constants of the plate are E, the modulus of elasticity and ν, Poisson's ratio. The plate is subjected to an eccentric force P acting normal to the plane of the plate at a point 0 located a distance $l = a\eta$ from $\bar{0}$ along the x-axis (Fig. 4).

Denoting the transverse displacement by $W(r, \theta)$, the governing equation is

$$\nabla^4 W(r,\theta) = \frac{P\delta(r)}{D} \tag{36}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{37a}$$

and

$$\nabla^4 = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta^2}\right)^2. \tag{37b}$$

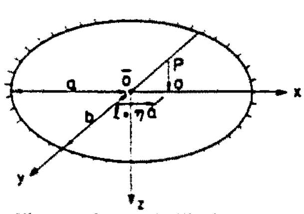


Fig. 4. Clamped elliptic plate.

The appropriate boundary conditions are then

$$W|_{C_e} = 0 \tag{38a}$$

$$\left.\frac{\partial W}{\partial n_e}\right|_{C_e} = 0 \tag{38b}$$

where n_e is the normal to C_e.

Following the previous section, $W(r,\theta)$ is postulated to be of the form given by eqn (5) where $W \equiv f$ here and below and it is noted that $D\nabla^4 \equiv L$ of Section 2.

From eqn (6a), the case $i = j = 0$ becomes

$$D\nabla^4 W^{(0,0)} = P\delta(r) \tag{39a}$$

subject to the conditions (see eqns (29))

$$W^{(0,0)} = 0, \quad W^{(0,0)}_{,r} = 0, \quad r = a. \tag{39b, c}$$

Equations (39), recognized as representing the axi-symmetric case of a clamped circular plate subjected to a central point load P, possess the known solution[5]

$$W^{(0,0)} = B\left[r^2 \log\left(\frac{r}{a}\right) + \frac{1}{2}(a^2 - r^2)\right] \tag{40a}$$

where

$$B = \frac{P}{8\pi D}. \tag{40b}$$

3.2. *The perturbed solutions*

Following eqn (6b), $W^{(i,j)}(r,\theta)$ must satisfy

$$\nabla^4 W^{(i,j)}(r,\theta) = 0, \quad i \geq 0, \quad j \geq 0, \quad (i+j > 0) \tag{41}$$

subject to the appropriate boundary conditions given by eqns (32) and (33) or eqns (A1)–(A8).

Suitable solutions of the biharmonic equation are of the form[6]

$$W^{(i,j)} = C_0 + C_1 r^2 + C_2 r^2 \log\left(\frac{r}{a}\right) + (\alpha_n r^n + \beta_n r^{n+2}) \cos n\theta \tag{42}$$

where C_0, C_1, C_2 and α_n, β_n ($n = 1, 2, 3, \ldots$) are constants.

The set of equations using the appropriate boundary conditions are now solved sequentially. The procedure for the first few cases is outlined.

3.2.1. *Case* $i = 0, j = 1$. Substituting, eqns (40) in eqns (32a) and (33a)

$$W^{(0,1)}|_{C_a} = 0, \quad W^{(0,1)}_{,r}|_{C_a} = Ba(1 - \cos 2\theta). \tag{43}$$

The equations are satisfied if $C_0 = -Ba^2/2$, $C_1 = B/2$, $\alpha_2 = B/2$, $\beta_2 = -B/2a^2$ and by setting the remaining constants appearing in eqn (42) to zero. Hence

$$W^{(0,1)}(r,\theta) = \frac{Ba^2}{2}\left[\left(\frac{r^2}{a^2}-1\right)\left(1-\frac{r^2}{a^2}\cos 2\theta\right)\right]. \quad (44)$$

3.2.2. *Case* $i = 0$, $j = 2$. Substituting, eqns (40) and (44) in the boundary conditions, eqns (A1a)–(A5a), and making suitable trigonometric substitutions, one obtains

$$W^{(0,2)}|_{C_a} = \frac{Ba^2}{8}(3-4\cos 2\theta+\cos 4\theta) \quad (45a)$$

$$W^{(0,2)}_{,r}|_{C_a} = \frac{3Ba}{4}(1-2\cos 2\theta+\cos 4\theta). \quad (45b)$$

By matching like terms at $r = a$, eqn (42) leads to

$$W^{(0,2)}(r,\theta) = \frac{B}{8}[3r^2-2r^2(1+r^2/a^2)\cos 2\theta+r^6/a^4\cos 4\theta]. \quad (46)$$

3.2.3. *Case* $i = 1$, $j = 0$. Substituting, eqns (40), (44) and (46) in eqns (32b) and (33b), one obtains, as before, for $r = a$

$$W^{(1,0)}|_{C_a} = 0 \quad (47a)$$

$$W^{(1,0)}_{,r}|_{C_a} = 2Ba\cos\theta. \quad (47b)$$

By a matching procedure, the constants of eqn (42) are immediately evaluated, leading to

$$W^{(1,0)}(r,\theta) = Ba^2\left(-\frac{r}{a}+\frac{r^3}{a^3}\right)\cos\theta. \quad (48)$$

The remaining set in the sequence of problems is solved similarly. Omitting all tedious algebraic details, the solutions are summarized below

$$W^{(1,1)}(r,\theta) = \frac{B}{2}\left[\left(3ar-\frac{2r^3}{a}\right)\cos\theta-\frac{r^5}{a^3}\cos 3\theta\right] \quad (49)$$

$$W^{(1,2)}(r,\theta) = \frac{B}{8}\left[2ar\left(-4+\frac{3r^2}{a^2}\right)\cos\theta+7r^2\left(\frac{r}{a}-\frac{r^3}{a^3}\right)\cos 3\theta+\frac{r^5}{a^3}\left(1+\frac{r^2}{a^2}\right)\cos 5\theta\right] \quad (50)$$

$$W^{(2,0)}(r,\theta) = \frac{B}{2}\left[-2a^2+3r^2+\frac{r^4}{a^2}\cos 2\theta\right] \quad (51)$$

$$W^{(2,1)}(r,\theta) = \frac{B}{2}\left[3(a^2-r^2)-r^2\left(2+\frac{r^2}{a^2}\right)\cos 2\theta-\frac{r^6}{a^4}\cos 4\theta\right] \quad (52)$$

$$W^{(2,2)} = \frac{B}{8}\left[2(-4a^2+5r^2)+r^2\left(31-25\frac{r^2}{a^2}\right)\cos 2\theta -6\frac{r^6}{a^4}\cos 4\theta+\frac{r^6}{a^4}\left(-9+11\frac{r^2}{a^2}\right)\cos 6\theta\right]. \quad (53)$$

Finally, then, the solution is given by

$$W(r,\theta) = \sum_{i=0}^{2}\sum_{j=0}^{2} \eta^i \varepsilon^j W^{(i,j)}(r,\theta) \tag{54}$$

where $W^{(i,j)}$ are given by eqns (40), (44), (46) and (48)–(53).

4. SOLUTION ALONG PRINCIPAL AXES. NUMERICAL RESULTS AND DISCUSSION

4.1. *Simplified expressions along principal axes*

Along the principal x- and y-axes, the derived expressions for $W(r,\theta)$ lead to considerable simplifications.

Defining the non-dimensional radial coordinate

$$\rho = \frac{r}{a} \tag{55a}$$

and letting

$$\xi = \frac{x}{a}, \quad -a < x < a \tag{55b}$$

$$\rho = \begin{cases} \xi-\eta, & \eta<\xi<1; \quad \theta = 0 \\ \eta-\xi, & -1<\xi<\eta; \quad \theta = \pi. \end{cases} \tag{56a}$$

Noting that

$$\rho^2 = (\xi-\eta)^2 \tag{56b}$$

and, in accordance with the second-order scheme developed here, upon keeping terms up to $O(\eta^2)$, expressions for $W^{(i,j)}$ appearing above become

$$W^{(0,0)} = Ba^2\{(\xi-\eta)^2 \log|\xi-\eta| + \tfrac{1}{2}[(1-\xi^2)+2\xi\eta-\eta^2]\} \tag{57a}$$

$$W^{(0,1)} = Ba^2\{-(1-\xi^2)[\tfrac{1}{2}(1-\xi^2)+2\xi\eta]+(1-3\xi^2)\eta^2\} \tag{57b}$$

$$W^{(0,2)} = \frac{Ba^2}{8}\{(1-\xi^2)[\xi^2(1-\xi^2)+2\xi(-1+3\xi^2)\eta]+(1-12\xi^2+15\xi^4)\eta^2\} \tag{57c}$$

$$W^{(1,0)} = Ba^2[\xi(\xi^2-1)+(1-3\xi^2)\eta+O(\eta^2)] \tag{57d}^\dagger$$

$$W^{(1,1)} = \frac{Ba^2}{2}\{\xi(3-2\xi^2-\xi^4)+(-3+6\xi^2+5\xi^4)\eta+O(\eta^2)\} \tag{57e}$$

$$W^{(1,2)} = \frac{Ba^2}{8}\{\xi(-8+13\xi^2-6\xi^4+\xi^6)+(8-39\xi^2+30\xi^4-7\xi^6)\eta+O(\eta^2)\} \tag{57f}$$

$$W^{(2,0)} = \frac{Ba^2}{2}\{-2+3\xi^2+\xi^4+O(\eta)\} \tag{57g}$$

$$W^{(2,1)} = \frac{Ba^2}{2}\{3-5\xi^2-\xi^4-\xi^6+O(\eta)\} \tag{57h}$$

$$W^{(2,2)} = \frac{Ba^2}{8}\{-6+41\xi^2-25\xi^4-15\xi^6+9\xi^8+O(\eta)\}. \tag{57i}$$

† It is noted that terms of order greater than $(2-i)$ appearing in expressions for $W^{(i,j)}$ do not contribute to solutions when using a second-order scheme.

Combining eqns (57) in accordance with eqn (54), recalling the expansion

$$\log(1 \pm X) = \pm X - \frac{X^2}{2} + \cdots, \quad |X| \ll 1 \tag{58}$$

for the case $|\eta| \ll |\xi|$ and collecting like powers of η and ε, yields finally

$$\begin{aligned}\left.\frac{W(\xi)}{Ba^2}\right|_{y=0} &= \xi^2 \log|\xi| + \tfrac{1}{2}(1-\xi^2) + \xi[\xi^2 - 1 - 2\log|\xi|]\eta - \tfrac{1}{2}(\xi^2-1)^2\varepsilon \\ &+ \tfrac{1}{2}[2\log|\xi| + (\xi^2-2)(\xi^2-1)]\eta^2 + \frac{\xi^2}{8}(\xi^2-1)^2\varepsilon^2 - \frac{\xi}{2}(1-\xi^2)^2\eta\varepsilon \\ &- \tfrac{1}{2}(\xi^2-1)^2(\xi^2-2)\eta^2\varepsilon + \frac{\xi}{8}(\xi^2-1)^2(\xi^2-10)\eta\varepsilon^2 + \frac{(\xi^2-1)^2}{8}(11\xi^4-4\xi^2+1)\eta^2\varepsilon^2.\end{aligned} \tag{59}$$

Along the y-axis, similar simplifications can be achieved. Defining the parameters

$$\zeta = \frac{y}{b} \tag{60a}$$

$$\chi = \frac{y}{a} = \frac{\zeta}{(1+\varepsilon)} \tag{60b}$$

noting, upon keeping terms up to $O(\eta^2)$, that

$$\rho^m \cos n\theta = (-1)^{n/2}[\chi^m - \tfrac{1}{2}(n^2-m)\chi^{m-2}\eta^2], \quad m, n \text{ even}; \quad m \geqslant n = 0, 2, 4, \ldots \tag{60c}$$

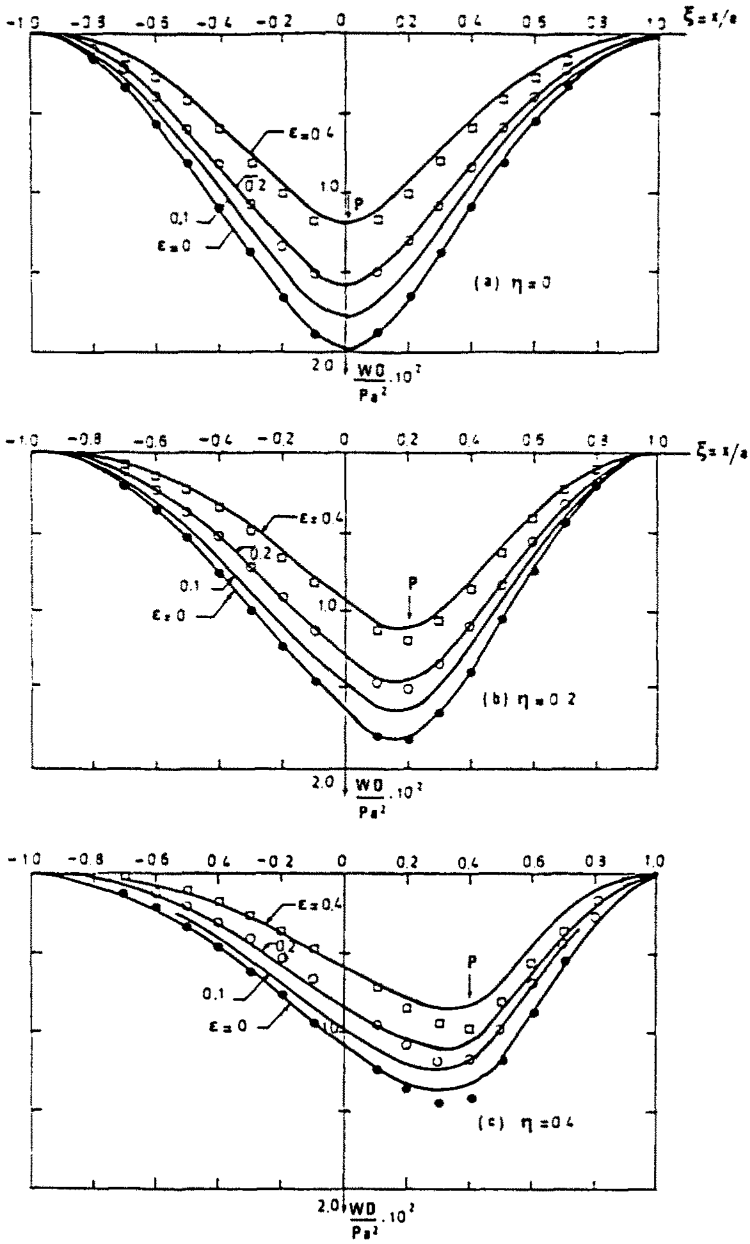
and

$$\rho^m \cos n\theta = -n(-1)^{(n-1)/2}\chi^{m-1}\eta, \quad m, n \text{ odd}; \quad m \geqslant n = 1, 3, 5, \ldots \tag{60d}$$

substituting in eqn (54), after expanding the logarithmic term and collecting terms in powers of ε and η, leads to

$$\begin{aligned}\left.\frac{W(\zeta)}{Ba^2}\right|_{x=0} &= \zeta^2 \log\zeta - \tfrac{1}{2}(\zeta^2-1) + [\tfrac{1}{2}(\zeta^2-1)(\zeta^2+1) - 2\zeta^2\log\zeta]\varepsilon \\ &+ \zeta^2[3\log\zeta + \tfrac{1}{8}(\zeta^2-1)(\zeta^2-13)]\varepsilon^2 + [\log\zeta - \zeta^2(\zeta^2-1)/2]\eta^2 - \frac{\zeta^2}{2}(\zeta^2-1)^2\eta^2\varepsilon \\ &- \tfrac{1}{8}(\zeta^2-1)^2(11\zeta^4-2\zeta^2-3)\eta^2\varepsilon^2.\end{aligned} \tag{61}$$

It is observed that, as opposed to eqn (59), eqn (61) is an even function of η, thus reflecting the symmetry of the problem.



F.E. values : ● ε = 0 ; ○ ε = 0.2 ; □ ε = 0.4

Fig. 5. Displacement distribution along the x-axis.

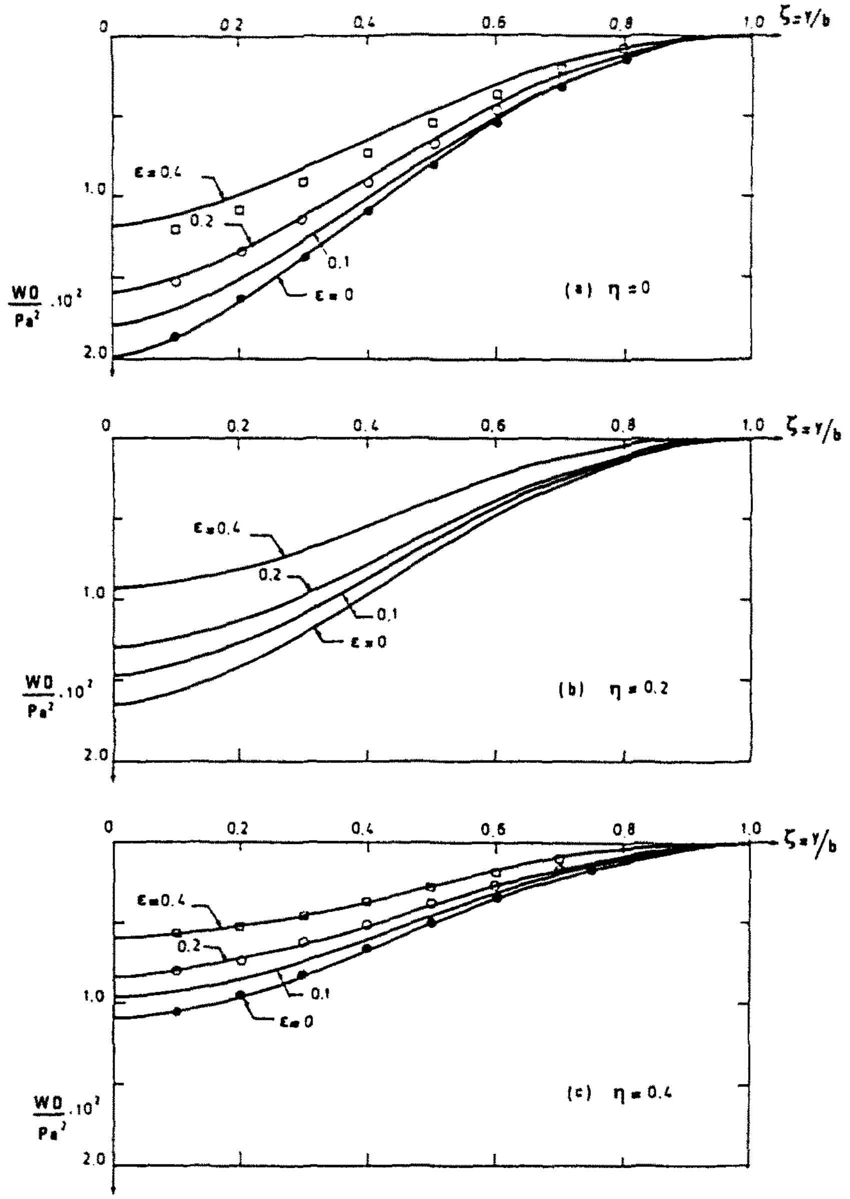
4.2. *Numerical results*

Numerical results for the normalized displacement, $(W/a)/(Pa/D)$, along the x-axis $(-1 \leqslant \xi = x/a \leqslant 1)$ are shown in Figs 5(a)–(c) for several loading positions: $\eta = 0, 0.2, 0.4$. In each case, the displacement is presented as a family of curves representing plates defined by ellipticity $\varepsilon = 0, 0.1, 0.2$ and 0.4.

Comparing among the figures one notes that the displacements decrease with increasing eccentricity of the load as well as with increasing ellipticity ε. It is observed that for all eccentric load positions, $\eta > 0$, the maximum displacement does not occur under the force but instead at an interior point, i.e. at $0 < \xi < \eta$.

The displacement variation along the y-axis with $\zeta = y/b$ is shown in Fig. 6 for $0 \leqslant |\zeta| \leqslant 1$. The maximum displacement is observed to occur at $\zeta = 0$ and it is noted that the displacements are considerably reduced with increasing eccentricity and ellipticity.

The effect of ellipticity and eccentricity is most readily seen in Fig. 7 where the normalized displacement of the centre point, W_0D/Pa^2, is presented by means of a family



F.E. values : ● ε = 0 ; ○ ε = 0.2 ; □ ε = 0.4

Fig. 6. Displacement distribution along the y-axis.

of curves for the various eccentricities as a function of ellipticity for $0 \leqslant \varepsilon \leqslant 0.4$, thus providing a quantitative description of the reduction of the displacement with ellipticity. It can be noted for example, that with $\varepsilon = 0.2$, the centre displacement of the ellipse with respect to that of a circle (the radius of which is the same as the semi-major radius of the ellipse) is reduced between 20% for a centrally applied load and by 24% for an eccentric load with $\eta = 0.4$; for $\varepsilon = 0.4$, the reduction is between 40 and 46%, respectively. In the case of a central load, $\eta = 0$, it is noted that the displacement W_0 is given by the simple expression

$$\frac{DW_0}{a^2} = \frac{P}{16\pi}(1-\varepsilon)+O(\varepsilon^3). \tag{62}$$

In Fig. 8, the displacement is shown by a family of curves representing the various ellipticities and as a function of the eccentricity parameter η. It is to be noted that the centre

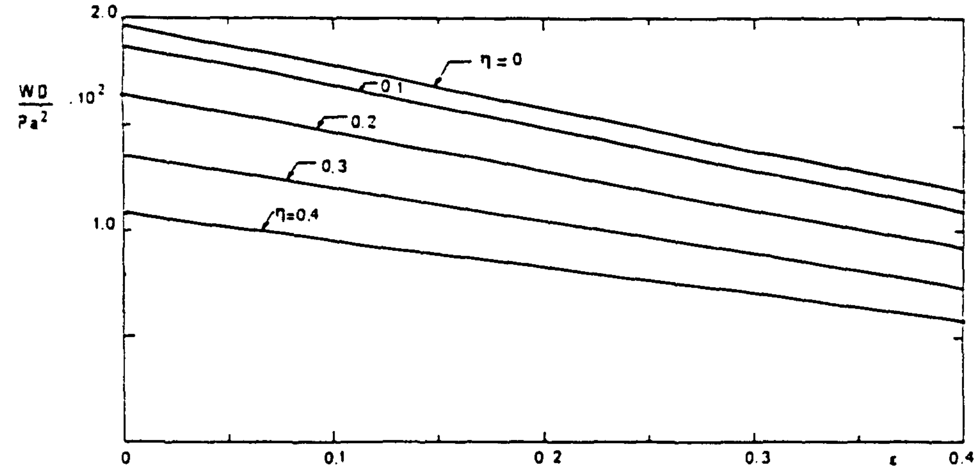


Fig. 7. Centre displacement vs ellipticity.

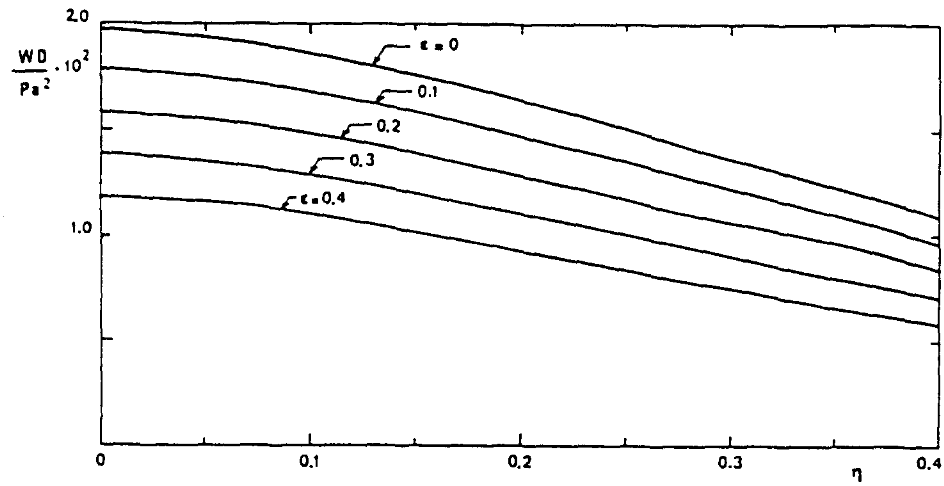


Fig. 8. Centre displacement vs eccentricity.

displacement W_0, according to eqn (59) or eqn (61) is a quadratic function of η, as reflected by the zero slopes of the curves at $\eta = 0$.

4.3. *Accuracy of solution and discussion*

Since there exists no previous solution to the present problem, in order to provide an indication of the accuracy of the BPM solution, displacements for specific values of ε and η were calculated by means of a finite element technique; these calculated values are shown in Figs 5 and 6.

It is noted that although the BPM and finite element solutions diverge with increasing values of ε and η, nevertheless the discrepancy between the two solutions is less than 10% even for values of $\varepsilon = 0.4$ and $\eta = 0.4$. (Results obtained for larger values of ε and η, e.g. $\varepsilon = \eta = 0.5$, reveal that for such values, the BPM no longer leads to solutions of admissible accuracy.)

From a comparison of the results, the BPM is seen to yield lower bounds for the displacements; such results are consistent with the conclusions given in Ref. [4], namely that the BPM yields upper bounds to the stiffness of elastostatic systems. Finally, it may be concluded that for moderate ellipticities and eccentricities, say $\varepsilon \leqslant 0.4$, $\eta \leqslant 0.4$, the BPM represents a relatively simple method of solution which leads to reasonably accurate results.

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